Pigeon Publishing

NSW HSC Syllabus

11

concept **mathematics** advanced

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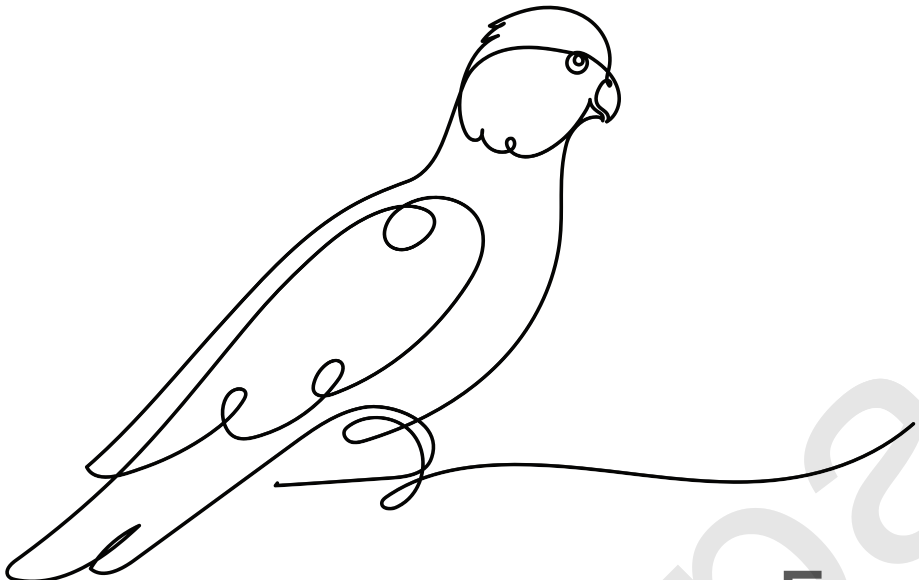
The Galah is a striking pink and grey cockatoo native to Australia, often seen in large, noisy flocks. It inhabits a wide range of environments, including woodlands, grasslands, and urban areas, feeding on seeds, nuts, and roots. Galahs are highly social and intelligent birds, known for their playful behaviour.



2 Functions

Functions provide a unified language for describing, interpreting and modelling relationships between varying quantities. This chapter introduces function notation, domain and range, and the algebraic and graphical conventions that connect multiple representations of the same rule. Emphasis is placed on recognising key families of functions, analysing their behaviour, and solving equations to model theoretical and real contexts.

2C

Finding Domain and Range

Finding domain and range graphically

Graphs provide a clear and intuitive way to interpret the domain and range of real functions, because they can be physically observed from the graph.

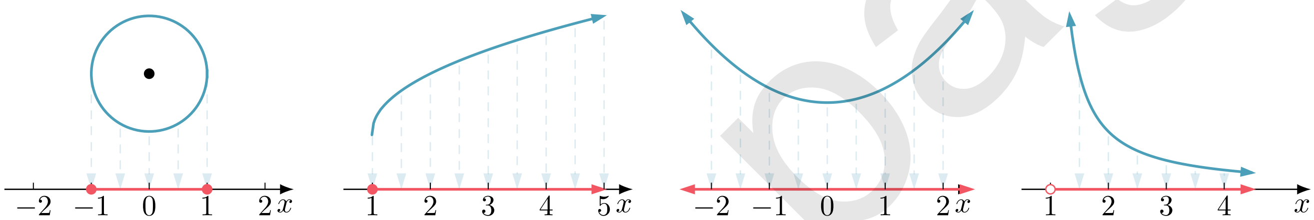
in short

The graph casts a shadow on the x-axis. The part the shadow covers is the domain.

Finding the domain from the graph

The domain of a function or relation can be determined visually by identifying all possible x-coordinates of the points on its graph.

This can be done by projecting (flattening) the curve onto the x-axis.



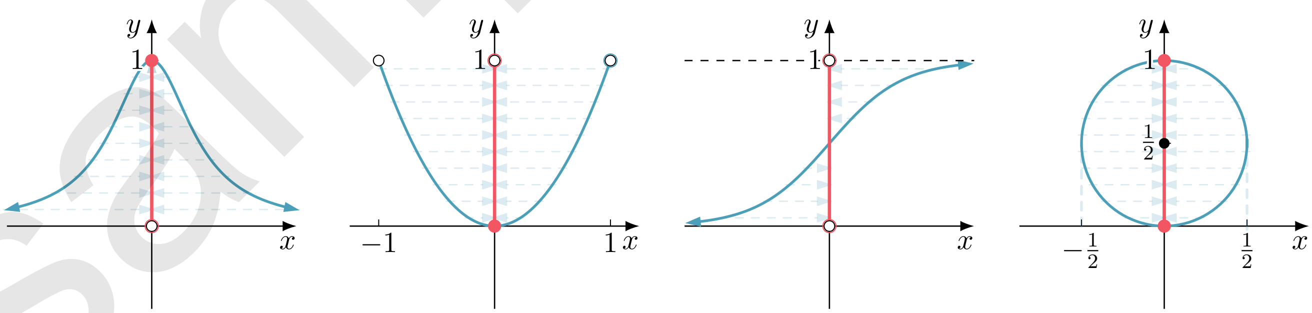
(a) $[-1, 1]$ (b) $[1, \infty)$ (c) $(-\infty, \infty)$ (d) $(1, \infty)$

Figure 8
The domain is the set of values that are covered by the projection of the curve onto the x-axis.

Finding the range from the graph

The range of a function or relation can be determined visually by identifying all possible y-coordinates of the points on its graph.

This can be done by projecting the curve onto the y-axis.



(a) $(0, 1]$ (b) $[0, 1)$ (c) $(0, 1)$ (d) $[0, 1]$

Figure 9
The range is the set of y-values that are covered by the projection.

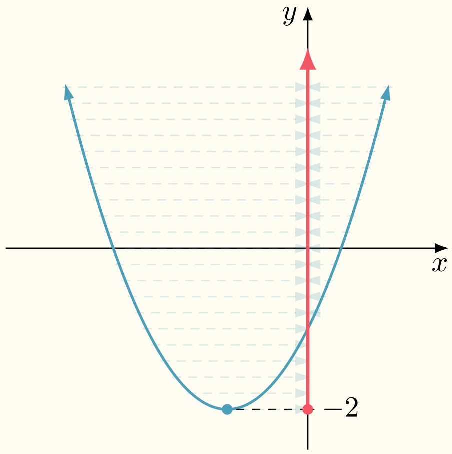
It is similar to the technique in finding the domain, but now with the y-axis instead.

example

Finding the range of a parabola

State the domain and range of $f(x) = x^2 + 2x - 1$.

solution



The graph of $y = f(x)$ is a concave-up parabola with vertex at $(-1, -2)$.

- The x-values can be anything, so the domain is $(-\infty, \infty)$.
- The y-values span from -2 upwards, so the range is $[-2, \infty)$.

Finding domain and range algebraically

When the graph is given:

- it is easier to find the domain and range of a function via the graph.

When the graph is *not* given, you may choose to either:

- sketch the graph, then find the domain and range graphically
- find the domain and range algebraically

We can find the domain of the following types of functions algebraically.

Square root functions

If the function is of the form $f(x) = \sqrt{g(x)}$, the domain can be found by solving $g(x) \geq 0$.

This is because we cannot take the square root of negative numbers, so the inside must either be positive or zero.

example *Domain of square root function*

Find the domain of $f(x) = \sqrt{x+1}$.

solution

The inside must be either positive or zero.

$$\begin{aligned} x + 1 &\geq 0 \\ x &\geq -1. \end{aligned}$$

Therefore, the domain is:

note to students

Pick one notation and stick to it. No need to write both.

Set Notation	$\{x \in \mathbb{R} \text{ where } x \geq -1\}$
Interval Notation	$[-1, \infty)$

example *Domain involving reversing the inequality sign*

Find the domain of $f(x) = \sqrt{4-2x}$.

solution

Similarly, let the inside be greater than or equal to zero, then solve.

$$\begin{aligned} 4 - 2x &\geq 0 \\ 4 &\geq 2x \\ x &\leq 2. \end{aligned}$$

Therefore, the domain is:

Set Notation	$\{x \in \mathbb{R} \text{ where } x \leq 2\}$
Interval Notation	$(-\infty, 2]$

Reciprocal functions

If the function is of the form $f(x) = \dfrac{k}{g(x)}$ for some constant k, the domain can be found by solving $g(x) \neq 0$.

This is because we cannot divide by zero, so the denominator can be anything other than zero.

example *Quadratic denominator*

Find the domain of the function $f(x) = \dfrac{1}{x^2 + 3x - 4}$.

solution

The domain can be found by solving:

$$\begin{aligned} x^2 + 3x - 4 &\neq 0 \\ (x+4)(x-1) &\neq 0 \\ x \neq -4 \text{ and } x &\neq 1. \end{aligned}$$

Therefore, the domain is:

Set Notation $\{x \in \mathbb{R} \text{ where } x \neq -4 \text{ and } x \neq 1\}$

Interval Notation $(-\infty, -4) \cup (-4, 1) \cup (1, \infty)$

Combination of the above

example *Square root on the denominator*

Find the domain of $f(x) = \dfrac{1}{\sqrt{1-x}}$.

solution

The domain can be found by combining the facts that division by zero is undefined and square roots must only take non-negative inputs, so:

$$\begin{aligned} 1 - x &> 0 \\ -x &> -1 \\ x &< 1 \end{aligned}$$

Therefore, the domain is:

Set Notation $\{x \in \mathbb{R} \text{ where } x < 1\}$

Interval Notation $(-\infty, 1)$

example *Combining different cases of a square root and denominator*

Find the domain of $f(x) = \dfrac{\sqrt{x}}{x-2}$.

solution

The numerator requires $x \geq 0$, but the denominator requires $x \neq 2$.

Therefore, the domain is:

Set Notation	$\{x \in \mathbb{R} \text{ where } x \geq 0 \text{ and } x \neq 2\}$
Interval Notation	$[0, 2) \cup (2, \infty)$

example *Combining different cases of two square roots*

Find the domain of $f(x) = \sqrt{x+4} + \sqrt{2-x}$.

solution

The left square root requires $x \geq -4$ but the left square root requires $x \leq 2$.

Therefore, the domain is:

Set Notation	$\{x \in \mathbb{R} \text{ where } -4 \leq x \leq 2\}$
Interval Notation	$[-4, 2]$

example *Combination of everything*

Find the domain of $f(x) = \sqrt{x+6} + \dfrac{1}{\sqrt{2-x}}$.

solution

why?

The inequality is strict because this square root is in the denominator, so $x \neq 2$.

The left square root requires $x \geq -6$ but the right square root requires $x < 2$.

Therefore, the domain is:

Set Notation	$\{x \in \mathbb{R} \text{ where } -6 \leq x < 2\}$
Interval Notation	$[-6, 2)$

Finding the range algebraically

note to students

The full extent of this method is shown in Composite Functions later in this chapter.

The range of the overall function is dictated by the range of the smaller functions inside.

- The range of a square root is non-negative i.e. $\sqrt{x} \geq 0$.
- The range of a square is non-negative i.e. $x^2 \geq 0$.
- The range of a reciprocal cannot be zero i.e. $\frac{1}{x} \neq 0$.

example

Range involving a square root

Find the range of $y = \sqrt{x+1} + 2$.

solution

Intuitive solution

- $\sqrt{x+1}$ is positive or zero.
- $y =$ (positive or zero) $+ 2$.

Therefore, the range is $y \geq 2$.

solution

Algebraic solution

$$\begin{aligned} \sqrt{x+1} &\geq 0 \\ \sqrt{x+1} + 2 &\geq 2 \\ y &\geq 2 \end{aligned}$$

Therefore, the range is $[2, \infty)$.

example

Range involving a squared term

Find the range of $y = 4 - x^2$.

solution

Intuitive solution

- x^2 is positive or zero.
- $y = 4 -$ (positive or zero).

Therefore, the range is $y \leq 4$.

solution

Algebraic solution

$$\begin{aligned} x^2 &\geq 0 \\ -x^2 &\leq 0 \\ 4 - x^2 &\leq 4 \\ y &\leq 4 \end{aligned}$$

Therefore, the range is $(-\infty, 4]$.

example

Range involving a reciprocal

Find the range of $y = \dfrac{2}{x} + 1$.

solution

Intuitive solution

- $\dfrac{2}{x}$ can be anything except zero.
- $y =$ (anything except zero) $+ 1$.
- $y =$ anything except 1.

Therefore, the range is $y > 1$ or $y < 1$.

solution

Algebraic solution

$$\begin{aligned} \frac{1}{x} &\neq 0 \\ \frac{2}{x} &\neq 0 \\ \frac{2}{x} + 1 &\neq 1 \\ y &\neq 1 \end{aligned}$$

Therefore, the range is $(-\infty, 1) \cup (1, \infty)$.

concept check

1 – 4 True or False? Let the domain of f be A and the domain of g be B.

1. The domain of $y = f(x) + g(x)$ is $A \cup B$.

2. The domain of $y = f(x) + g(x)$ is $A \cap B$.

3. The domain of $y = \sqrt{f(x)}$ is A.

4. The domain of $y = \dfrac{1}{g(x)}$ is B.

---

5. Match each of the functions below with their range.

(I) Range $= [0, \infty)$

(II) Range $= (0, \infty)$

(III) Range $= (-\infty, 0) \cup (0, \infty)$

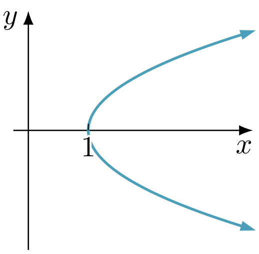
(IV) Range $= (-\infty, \infty)$

(a) $y = 2x - 1$ (b) $y = \sqrt{x}$

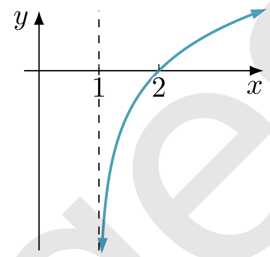
(c) $y = x^2$ (d) $y = x^3$

(e) $y = \dfrac{1}{x}$ (f) $y = \dfrac{1}{x^2}$

6. Match the following graphs with their correct domain and ranges.



(I)



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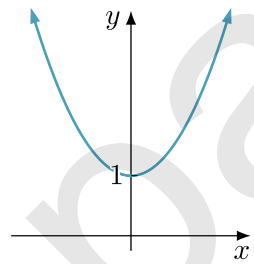

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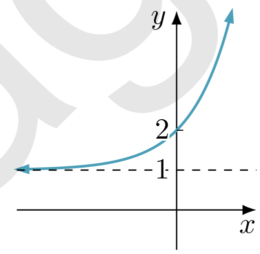

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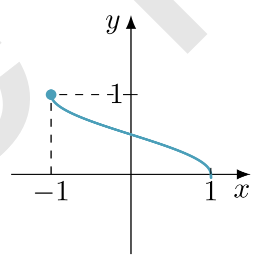
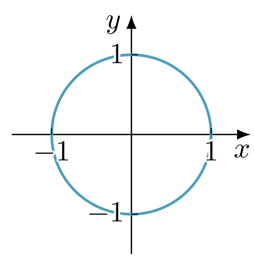


(V)



(VI)

(a) Domain $= (-\infty, \infty)$
Range $= [1, \infty)$

(b) Domain $= (-\infty, \infty)$
Range $= (1, \infty)$

(c) Domain $= [-1, 1]$
Range $= [-1, 1]$

(d) Domain $= (1, \infty)$
Range $= (-\infty, \infty)$

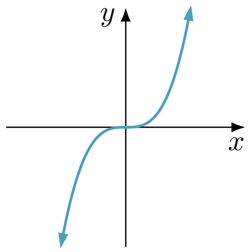
(e) Domain $= [1, \infty)$
Range $= (-\infty, \infty)$

(f) Domain $= [-1, 1]$
Range $= [0, 1]$

essential problems

7–18 State the domain and range of the following.

7.



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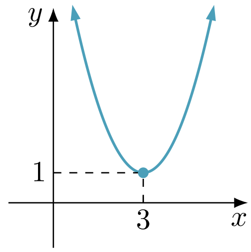

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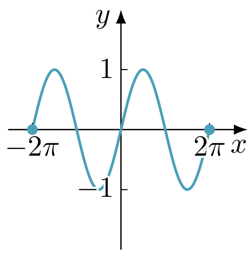

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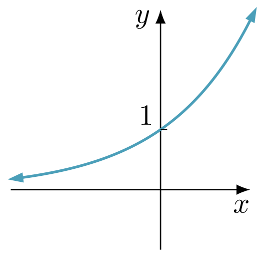

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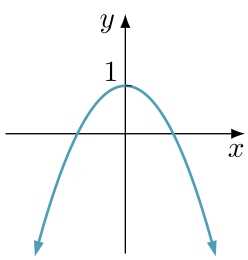

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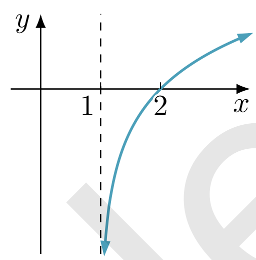

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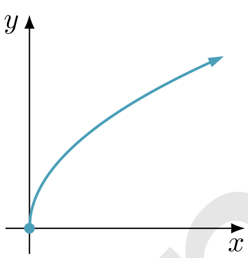

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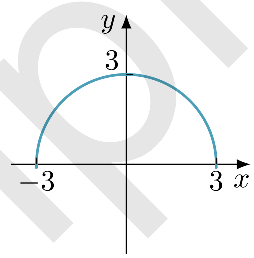

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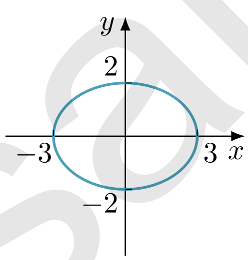

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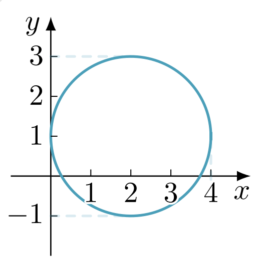

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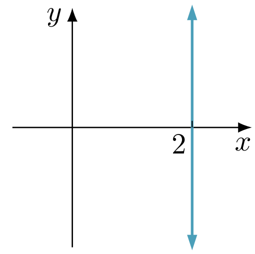

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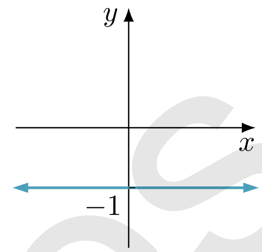


19–26 Find the domain of the following functions.

19. $f(x) = 2x + 3$

20. $f(x) = \sqrt{x}$

21. $f(x) = \dfrac{1}{x}$

22. $f(x) = 1 + \dfrac{3}{x+4}$

23. $f(x) = 1 + x^2$

24. $f(x) = \dfrac{1}{x^2 - 4}$

25. $f(x) = \sqrt{x-2}$

26. $f(x) = \dfrac{1}{\sqrt{2x-1}}$

27–32 Find the range of the following functions.

27. $f(x) = 3x + 1$

28. $f(x) = x^2 - 1$

29. $f(x) = x^2 + 3$

30. $f(x) = 4x^2 - 3$

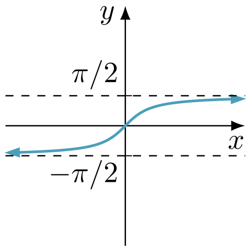
31. $f(x) = 4 - 3x^2$

32. $f(x) = x^3 - 2$

additional practice

33 – 40 State the domain and range of the following.

33.



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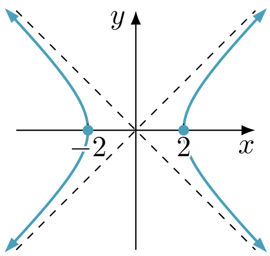

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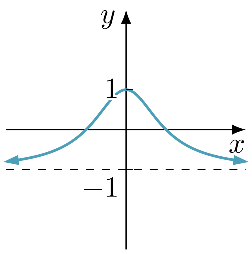

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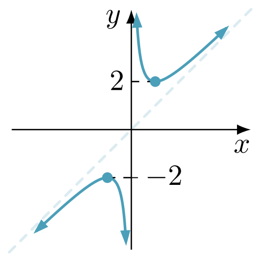

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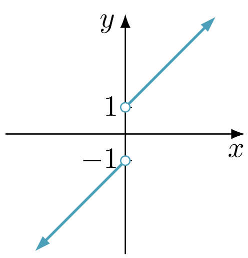

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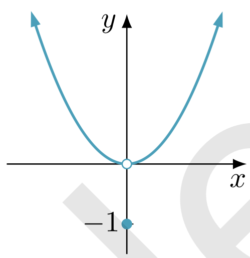

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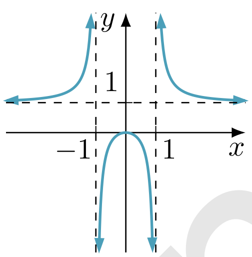

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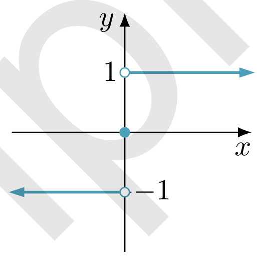

41 – 48 Find the domain of the functions below.

41. $f(x) = \dfrac{x+1}{x-2}$

42. $f(x) = \dfrac{1}{x^2+2x-3}$

43. $f(x) = \sqrt{4-x}$

44. $f(x) = \sqrt{2x+6}$

45. $f(x) = \sqrt[3]{x}$

46. $f(x) = \sqrt{x^2+1}$

47. $f(x) = \dfrac{1}{\sqrt{2-x}}$

48. $f(x) = \dfrac{x}{\sqrt[3]{x+4}}$

49. Find the domain of $y = \dfrac{\sqrt{x-1}}{x-4}$.

50. Find the domain of $f(x) = \dfrac{1}{1+\frac{1}{x+1}}$.

51. Find the domain of

$$f(x) = \frac{\sqrt{2-x}+\sqrt{2+x}}{4-x^2}.$$

52. Find the domain of $y = \sqrt{\dfrac{x+1}{x-1}}$.
Hint: If $a/b > 0$, then both $a, b > 0$, or both $a, b < 0$.

53 – 60 Find the range of the following functions.

53. $f(x) = \dfrac{1}{x} + 2$

54. $f(x) = \dfrac{2}{x-1} + 3$

55. $f(x) = \dfrac{1}{x^2} - 5$

56. $f(x) = -\dfrac{5}{x^2} + 1$

57. $f(x) = \dfrac{x^2+4}{2}$

58. $f(x) = \sqrt{x+1}$

59. $f(x) = \sqrt{x} - 2$

60. $f(x) = 4 - \sqrt{x-1}$

61. Find the domain and range of $y = \dfrac{\sqrt{x-1}}{\sqrt{4-x}}$.

62. Find the domain and range of $y = \dfrac{\sqrt{x^2-1}}{\sqrt{x+1}}$.

63. Find the domain and range of $y = x + \sqrt{x}$.

64 – 65 Find the domain and range of the following functions.

64. $f(x) = \sqrt{x} + \sqrt{4-x}$

65. $f(x) = \dfrac{1}{\sqrt{x}} + \dfrac{1}{\sqrt{8-x}}$

66. Find the domain and range of
$f(x) = \dfrac{4}{2-\sqrt{3-x}}$.